**Nonlinear Kinematic Wave Verification**

A second verification analysis is presented to demonstrate the GEM flexibility to accommodate and solve PDEs that do not strictly follow the format presented previously as equation (4). Here, we the GEM default mass balance equation by a suitable redefinition of SVs and parameters to yield the nonlinear kinematic wave equation[[1]](#footnote-1), which describes transient surface water flow change with respect to time and distance along a stream channel. Nominally, the GEM is designed to simulate chemical F&T given user-specified flows in space and time. This analysis models the transient flows themselves. The analysis also demonstrates the GEM’s nonlinear functionality.

The kinematic wave equation represents a joint solution of the continuity equation

 (1)

where

*Q =* volumetric flow rate (L3/T), a function of time and distance, i.e. Q(x,t)

*A =* flow cross-sectional area (L2), a function of time and distance, i.e. A(x,t)

q(t) = lateral inflow per unit length (L2/T), a function of time, i.e. q(t)

and the momentum equation, expressed as:

 (2)

where

*α* and *β* = stream channel parameters

Substituting (2) into (1) results in the combined equation (Chow et al., 1988)

 (3)

Chow et al. (1988) solve equation (3) in their Example 9.6.1 (page 297) using a FTBS finite difference scheme. Their FTBS-discretized equation is, for node “i” at time “t from Chow et al. (1988), pg. 300:

 (4)

Chow et al. solved the discretized kinematic wave equation (4) in an example for a 200-foot-wide rectangular channel that is 15,000 feet long, with a bed slope of 1 percent and a Manning’s roughness factor of 0.035. The solution routed an inflow hydrograph through the example channel for 150 minutes.

We set up an 8-compartment, 1-D GEM model to represent the rectangular channel. The compartments are identically sized and 3,000 feet long. Compartments 1 and 8 are boundary compartments (N=6) with constant flow of 2,000 cfs, corresponding to the Chow et al. example. By simply reinterpreting the GEM’s concentration SV -- “Cd” in equation (4) as the flow rate “Q” in equation (4) -- and making other parameter adjustments, the underlying GEM PDE is readily configured to represent equation (4). Details of this manipulation are provided in Appendix A for the interested reader.

We solved the resulting (nonlinear) GEM-based equation using the GEM’s quasi-Newton method, using the same time step as Chow et al. of 180 seconds and ran the routing simulation beyond 10,000 seconds (150 minutes). Figure 3 shows a comparison of the GEM results with those from Chow et al., matching the GEM compartments to the corresponding Chow et al. distances. Chow et al. only show a portion of their complete time series results so there are gaps in those time series in Figure 1.

As can be seen, the results between the GEM and Chow et al. are essentially identical. This confirms the applicability of the GEM to solving the nonlinear, kinematic wave equation as well as providing an additional verification example.

**Figure 1. Comparison of GEM and Chow et al. Results for Nonlinear Kinematic Wave Equation**

**References**

V.T. Chow, Maidment, D.R., and Mays, L.W., Applied Hydrology, McGraw-Hill, Inc., New York, 1988.

**Appendix**

**Conversion of the GEM PDE to Represent the Chow et al. (1988) Nonlinear, Combined Continuity and Momentum Equation**

Re-expressing the Chow et al. equation (4), repeated here:

 (A-1)

in a GEM-like format (using R) gives

 (A-2a)

where

 (A-2b)

Using a FTBS approach, the GEM mass balance equation (4) can be written in 1-D as

 (A-3)

where no sources/sinks are included, the compartments’ geometry (volume, area) are assumed to be static in time, and

Wit = an external loading to compartment i at time t with units of M/T

 = interfacial flow volume (L3/T) between compartments i and j. (Signs on flows are explicitly included in equation (19)).

 = interfacial area (L2) between compartments i and j

= volume of compartment i (L3)

Ei,j = dispersion coefficient (L2/T) between compartments i and j

Li,j = length over which dispersive mixing occurs between compartments i and j (L)

Θi = water content (L3/L3) of medium

A note about  in equation (A-3) versus “Q” in equation (4) is in order. Q (and A) are SVs of the kinematic wave equation. Their solution in space and time is to be determined.  (and and ) in the GEM equation (4) are parameters of the equation. Their values are assumed to be known. We are trying to parameterize ,, and  in equation (18) in such a way that, along with other changes, the GEM compartment equation (A-3) becomes equivalent to Chow’s finite difference equation (A-1). The overbars on ,, and in equation (A-3) are being used here to remind the reader of that difference. That is, Q, A, and V will refer to the true SV while ,, and will remain simply GEM parameters.

To configure the GEM compartment equation



to represent the FTBS numerical representation of the kinetic wave partial differential equation (4), we assign:

Ei,j = 0

Θi = 1

Assume the GEM geometry parameters, and , do not change along the modeled length or with time. Also, assign = , i.e. the GEM’s flow parameter is normalized to equal the cross-sectional area. (The flow velocity is then unity.)

Substitute Δx for volume,, and assign = 1.

With these assignments, equation (A-3) becomes



which, after dividing through by Δx is:

 (A-4a)

where,  (A-4b)

If one makes a simple change-of-variables in (A-4) such that the GEM SV “Cd” is identically equal to the kinematic wave SV, Q, comparing the GEM equation (A-4) with the FTBS finite difference equation for the kinetic wave equation (A-2), shows that they are essentially identical. The only exception is the trivial difference that the distributed inflow term, , is based on an average inflow over time steps t and t+1, while our distributed “inflow” term  occurs all within the current time step t. The GEM policy is for time step “t+1” to represent the first instant of the upcoming time step, not sometime *within* (e.g., midpoint) t+1. With this definition, the GEM inflow parameter makes sense.

A note on units is useful. The GEM assumes lengths are in meters, mass in grams, time in days, etc. and the GEM input files contain these default units in the header records. However, no unit conversions are performed anywhere in the GEM code. Therefore, the user may populate the input files using whatever units they desire, so long as the units are internally consistent in the resulting equations. For example, the Chow et al. example uses feet and seconds instead of meters and days. In this application, we simply used the same units in the GEM files as Chow et al.

1. In the GEM’s Equation Solver Mode, one would simply enter the finite difference equations themselves and no “redefinition of SVs and parameters” is necessary. This approach would be completely generic, analogous to using commercial equation solvers such as MatLab or Mathematica. However, the Environmental System Mode is intended to facilitate environmental numerical modeling and we are here exploring the ability to simulate environmental systems other than chemical mass F&T using Environmental System Mode. [↑](#footnote-ref-1)